New set of symmetries of the integrable equations, Lie algebra and non-isospectral evolution equations. II. AKNS system

1986 J. Phys. A: Math. Gen. 19 3713

(http://iopscience.iop.org/0305-4470/19/18/019)

J. Phys. A: Math. Gen. 19 (1986) 3713-3725. Printed in Great Britain

New set of symmetries of the integrable equations, Lie algebra and non-isospectral evolution equations: II. AKNS system

Li Yi-shen†‡ and Zhu Guo-cheng§

† International Centre for Theoretical Physics, Miramare, Trieste, Italy
§ Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, People's Republic of China

Received 15 January 1986, in final form 19 March 1986

Abstract. In this paper, we define directly a new set of symmetries for the AKNS system and prove that they constitute an infinite-dimensional Lie algebra with the 'old' symmetries. We also point out the relation between the new symmetry and the non-isospectral problem.

We use the reduction technique and point out that the NLS, MKdV, SG and sinh-G hierarchies have two sets of symmetries which constitute an infinite-dimensional Lie algebra. These results are extensions of those of Chen *et al.*

1. Introduction

It is well known that integrable non-linear evolution equations in general possess an infinite number of classical conservation laws. Associated with these conservation laws are classical symmetries (called k symmetries) which do not depend explicitly on the space and time variables.

Recently, new symmetries (called τ symmetries) which depend on the space and time variables explicitly have been found. Olver (1980) found a new symmetry for the KdV equation, Fokas and Fuchssteiner (1981) found part of τ_2 for the BO equation and used this part of τ_2 to derive the classical symmetry K_n. Then Oevel and Fuchssteiner (1982) found part of τ_3 for the KP equation and used this part of τ_3 to derive the classical symmetries. At the same time, Chen *et al* (1982a) found the new hierarchy of symmetries for six non-linear evolution equations: the KdV, MKdV, NLS, SG, BO and KP equations. These new symmetries were defined recursively through the Lie product of a specially chosen new symmetry with previously obtained ones. Together with the known classical symmetries, they constitute an infinite-dimensional Lie algebra.

We found that the first new symmetry of many evolution equations appeared naturally in the corresponding evolution equations which related to the non-isospectral problem. In a previous paper (Li and Zhu 1985) we defined directly the new set of symmetries for the KdV equation by a recursion operator and then proved that they constituted an infinite Lie algebra with classical symmetries. In this paper, we extend this to the AKNS system.

This paper is organised as follows. In § 2 we introduce some notation and verify that the recursion operator ϕ (see (2.2)) is a hereditary symmetry and is a strong

‡ On leave of absence from Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.

symmetry of the corresponding evolution equation and then we prove some lemmas. In § 3, we prove that there are two sets of symmetries of the AKNS hierarchy and that they constitute an infinite-dimensional Lie algebra. In § 4 we consider the symmetries and Lie algebras of NLS, MKdV, SG and sinh-Gordon hierarchies by a reduction technique and obtain more symmetries than in Chen *et al* (1982a) for the last three hierarchies. Finally we give an explicit formula for the inverse operator of ϕ_1 (see (4.1*b*)) in the appendix.

2. Notation and some lemmas

Let U be a vector space constructed by all vector functions $f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, where f_1 and f_2 are functions of x and t and assume that f_1 and f_2 are sufficiently smooth functions, i.e. f_1, f_2 and their derivatives of any possible order with respect to t and x vanish rapidly as $|x|\to\infty$.

We always assume $u=\begin{pmatrix} q \\ r \end{pmatrix}\in U$ in the following. Set

$$\boldsymbol{u}=\begin{pmatrix} -\mathrm{i}q \\ \mathrm{i}r \end{pmatrix}$$

and consider the AKNS hierarchy

$$u_t=k_m=\phi^m\boldsymbol{u} \qquad m=0,1,2,\ldots \tag{2.1}$$

where

$$\phi=\frac{1}{\mathrm{i}}\begin{pmatrix} -D+2qD^{-1}r & 2qD^{-1} \\ -2rD^{-1}r & D-2rD^{-1}q \end{pmatrix} \tag{2.2}$$

$$D=\partial/\partial x \qquad D^{-1}D=DD^{-1}=1 \qquad D^{-1}=\int_{-\infty}^{x}\mathrm{d}x$$

(or take $I=\int_x^{+\infty}\cdot\,\mathrm{d}x$ instead of $-D^{-1}=-\int_{-\infty}^{x}\cdot\,\mathrm{d}x$ in (2.2)).

The first three k_i are

$$k_0=\begin{pmatrix} -\mathrm{i}q \\ \mathrm{i}r \end{pmatrix} \qquad k_1=\phi k_0=\begin{pmatrix} q_x \\ r_x \end{pmatrix} \qquad k_2=\phi^2k_0=\frac{1}{\mathrm{i}}\begin{pmatrix} -q_{xx}+2q^2r \\ r_{xx}-2r^2q \end{pmatrix}. \tag{2.3}$$

Set $G(u)=G(q,q_x,\ldots,r,r_x,\ldots)$ where G can be a function or operator. We define the Gateaux derivative

$$G'(u)[\sigma]=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(u+\varepsilon\sigma)\bigg|_{\varepsilon=0}. \tag{2.4}$$

$G'(u)[\sigma]$ is the derivative of G in the direction $\sigma=\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ at the point $u=\begin{pmatrix} q \\ r \end{pmatrix}$.

We have the following formula by definition:

$$(\phi G)'[\sigma]=\phi'[\sigma]G+\phi G'[\sigma]. \tag{2.5}$$

Consider the linearised equation of (2.1):

$$\sigma_t=K'_m[\sigma]. \tag{2.6}$$

We write first three K'_m:

$$K'_0=\mathrm{i}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\qquad K'_1=\begin{pmatrix}D & 0\\ 0 & D\end{pmatrix}\qquad K'_2=\frac{1}{\mathrm{i}}\begin{pmatrix}-D^2+4qr & 2q^2\\ -2r^2 & D^2-4rq\end{pmatrix}. \tag{2.7}$$

A solution of (2.6), $\sigma=\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}$, is called a symmetry of (2.1).

We now prove the following two lemmas for the operator ϕ.

Lemma 1.

$$\phi'[x\boldsymbol{u}]+\phi(x\boldsymbol{u})'-(x\boldsymbol{u})'\phi=I \tag{2.8}$$

where $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Proof. It is easy to verify that

$$\phi'[x\boldsymbol{u}]=\begin{pmatrix}2qD^{-1}xr-2xqD^{-1}r & -2qD^{-1}xq-2xqD^{-1}q\\ -2rD^{-1}xr-2xrD^{-1}r & 2rD^{-1}xq-2xrD^{-1}q\end{pmatrix}$$

$$\phi(x\boldsymbol{u})'=\phi\begin{pmatrix}-\mathrm{i}x & 0\\ 0 & \mathrm{i}x\end{pmatrix}=\begin{pmatrix}1-2qD^{-1}xr & 2qD^{-1}xq\\ 2rD^{-1}xr & 1-2rD^{-1}xq\end{pmatrix}+\begin{pmatrix}xD & 0\\ 0 & xD\end{pmatrix}$$

$$(x\boldsymbol{u})'\phi=\begin{pmatrix}-\mathrm{i}x & 0\\ 0 & \mathrm{i}x\end{pmatrix}\phi=\begin{pmatrix}xD+2xqD^{-1}r & -2xqD^{-1}q\\ -2xrD^{-1}r & xD-2xrD^{-1}q\end{pmatrix}.$$

From the above, we get (2.8) immediately.

Lemma 2.

$$\phi'[\phi f]g-\phi'[\phi g]f=\phi\{\phi'[f]g-\phi'[g]f\} \tag{2.9}$$

where f and g are arbitrary functions in U. An operator with this property is called a hereditary symmetry (see Fuchssteiner and Fokas 1981).

Proof. For any $f=\begin{pmatrix}f_1\\ f_2\end{pmatrix}$ and $g=\begin{pmatrix}g_1\\ g_2\end{pmatrix}$, we have

$$\phi'[f]g=\frac{2}{\mathrm{i}}\begin{pmatrix}qD^{-1}f_2g_1+f_1D^{-1}rg_1+qD^{-1}f_1g_2+f_1D^{-1}qg_2\\ -rD^{-1}f_2g_1-f_2D^{-1}rg_1-rD^{-1}f_1g_2-f_2D^{-1}qg_2\end{pmatrix}.$$

This yields

$$\phi'[f]g-\phi'[g]f=\frac{2}{\mathrm{i}}\begin{pmatrix}f_1D^{-1}(rg_1+qg_2)-g_1D^{-1}(rf_1+gf_2)\\ -f_2D^{-1}(rg_1+qg_2)+g_2D^{-1}(rf_1+qf_2)\end{pmatrix}.$$

Thus

$$\begin{aligned}
&\phi\{\phi'[f]g-\phi'[g]f\}\\
&\quad=-2\begin{pmatrix}-f_{1x}D^{-1}(rg_1+qg_2)+g_{1x}D^{-1}(rf_1+qf_2)\\ -f_{2x}D^{-1}(rg_1+qg_2)+g_{2x}D^{-1}(rf_1+qf_2)\end{pmatrix}-2\begin{pmatrix}-qf_1g_2+qg_1f_2\\ -rf_2g_1+rg_2f_1\end{pmatrix}\\
&\qquad-4\begin{pmatrix}qD^{-1}r[f_1D^{-1}(rg_1+qg_2)-g_1D^{-1}(rf_1+gf_2)]\\ \qquad+qD^{-1}q[-f_2D^{-1}(rg_1+qg_2)+g_2D^{-1}(rf_1+qf_2)]\\ -rD^{-1}r[f_1D^{-1}(rg_1+qg_2)-g_1D^{-1}(rf_1+gf_2)]\\ \qquad-rD^{-1}q[-f_2D^{-1}(rg_1+qg_2)+g_2D^{-1}(rf_1+qf_2)]\end{pmatrix}.
\end{aligned}$$

Put

$$\phi f=\frac{1}{\mathrm{i}}\begin{pmatrix}-f_{1x}+2qD^{-1}(rf_1+qf_2)\\ f_{2x}-2rD^{-1}(rf_1+qf_2)\end{pmatrix}=\frac{1}{\mathrm{i}}\begin{pmatrix}s_1\\ s_2\end{pmatrix}$$

then

$$\begin{aligned}\phi'[\phi f]g&=-2\begin{pmatrix}qD^{-1}s_2+s_1D^{-1}r & qD^{-1}s_1+s_1D^{-1}q\\ -rD^{-1}s_2-s_2D^{-1}r & -rD^{-1}s_1-s_2D^{-1}q\end{pmatrix}\begin{pmatrix}g_1\\ g_2\end{pmatrix}\\ &=-2\begin{pmatrix}-f_{1x}D^{-1}(rg_1+qg_2)+qD^{-1}f_{2x}g_1-qD^{-1}f_{1x}g_2\\ -f_{2x}D^{-1}(rg_1+qg_2)-rD^{-1}f_{2x}g_1+rD^{-1}f_{1x}g_2\end{pmatrix}\\ &\quad-4\begin{pmatrix}qD^{-1}(rf_1+qf_2)D^{-1}(rg_1+qg_2)+qD^{-1}(-rg_1+qg_2)D^{-1}(rf_1+qf_2)\\ rD^{-1}(rf_1+qf_2)D^{-1}(rg_1+qg_2)+rD^{-1}(rg_1-qg_2)D^{-1}(rf_1+qf_2)\end{pmatrix}.\end{aligned}$$

Thus

$$\begin{aligned}&\phi'[\phi f]g-\phi'[\phi g]f\\ &\quad=-2\begin{pmatrix}-f_{1x}D^{-1}(rg_1+qg_2)+g_{1x}D^{-1}(rf_1+qf_2)\\ -f_{2x}D^{-1}(rg_1+qg_2)+g_{2x}D^{-1}(rf_1+qf_2)\end{pmatrix}-2\begin{pmatrix}qf_2g_1-qf_1g_2\\ -rf_2g_1+rf_1g_2\end{pmatrix}\\ &\quad-4\begin{pmatrix}qD^{-1}(-rg_1+qg_2)D^{-1}(rf_1+qf_2)-qD^{-1}(-rf_1+qf_2)D^{-1}(rg_1+qg_2)\\ rD^{-1}(rg_1-qg_2)D^{-1}(rf_1+qf_2)-rD^{-1}(rf_1-gf_2)D^{-1}(rg_1+qg_2)\end{pmatrix}.\end{aligned}$$

From the above, we obtain

$$\phi'[\phi f]g-\phi'[\phi g]f=\phi\{\phi[f]g-\phi'[g]f\}.$$

Lemma 3.

$$\phi'[K_m]=[K'_m,\phi]. \tag{2.10}$$

This shows that ϕ is a strong symmetry for equation (2.1).

Proof. Set $u_t=K_0$, i.e. $\begin{pmatrix}q_t\\ r_t\end{pmatrix}=\begin{pmatrix}-\mathrm{i}q\\ \mathrm{i}r\end{pmatrix}$.

We verify $\phi'[K_0]=[K'_0,\phi]$. It is easy to verify that

$$\phi'[K_0]=-4\begin{pmatrix}0 & qD^{-1}q\\ rD^{-1}r & 0\end{pmatrix}\qquad [K'_0,\phi]=\begin{pmatrix}0 & -4qD^{-1}q\\ -4rD^{-1}r & 0\end{pmatrix}.$$

Thus we obtain $\phi'[K_0]=[K'_0,\phi]$.

Since ϕ is a hereditary symmetry and a strong symmetry for $u_t=K_0$, we have that ϕ is a strong symmetry for $u_t=\phi^mK_0=K_m$. Hence (2.10) holds.

Define the Lie product as follows:

$$[K,G]=K'[G]-G'[K]. \tag{2.11}$$

Now we prove the following lemmas.

Lemma 4.

$$[K_m,K_n]=K'_m[K_n]=K'_n[K_m]=0\qquad m,n=0,1,2,\ldots. \tag{2.12}$$

Proof. Using the property of the strong symmetry of the operator ϕ and (2.5), we have

$$\begin{aligned}
[K_m, K_n] &= (\phi^m u)'[\varphi^n u] - (\phi^n u)'[\phi^m u] \\
&= \phi'[\phi^n u]\phi^{m-1}u + \phi(\phi^{m-1}u)'[\phi^n u] - (\phi^n u)'[\phi^m u] \\
&= [(\phi^n u)', \phi]\phi^{m-1}u + \phi(\phi^{m-1}u)'[\phi^n u] - (\phi^n u)'[\phi^m u] \\
&= (\phi^n u)'\phi^m u - \phi(\phi^n u)'[\phi^{m-1}u] + \phi(\phi^{m-1}u)'[\phi^n u] - (\phi^n u)'[\phi^m u] \\
&= \phi[K_{m-1}, K_n]. \qquad (2.13)
\end{aligned}$$

Then we can obtain (2.12) from (2.13) by induction.

Lemma 5.

$$[\phi^m u, xu] = m\phi^{m-1}u = mK_{m-1} \qquad m = 1, 2, \ldots . \qquad (2.14)$$

Proof. When $m = 1$

$$[\phi u, xu] = (\phi u)'[xu] - (xu)'[\phi u]$$

$$= \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} -\mathrm{i}xq \\ \mathrm{i}xr \end{pmatrix} - \begin{pmatrix} -\mathrm{i}x & 0 \\ 0 & \mathrm{i}x \end{pmatrix}\begin{pmatrix} q_x \\ r_x \end{pmatrix} = \begin{pmatrix} -\mathrm{i}D(xq) + \mathrm{i}xq_x \\ \mathrm{i}D(xr) - \mathrm{i}xr_x \end{pmatrix} = \begin{pmatrix} -\mathrm{i}q \\ \mathrm{i}r \end{pmatrix} = u = K_0.$$

(2.14) holds when $m = 1$ and assume that

$$[\phi^{m-1}u, xu] = (m-1)K_{m-2} = (m-1)^{m-2}u.$$

Then

$$\begin{aligned}
[\phi^m u, xu] &= \phi'[xu]\phi^{m-1}u + \phi(\phi^{m-1}u)'[xu] - (xu)'[\phi^m u] \\
&= \phi'[xu]\phi^{m-1}u + \phi[(\phi^{m-1}\mu)'[xu] \\
&\quad - (xu)'\phi^{m-1}u + (xu)'[\phi^{m-1}u] - (xu)'[\phi^m u] \\
&= \phi[\phi^{m-1}u, xu] + \{\phi'(xu) + \phi(xu)' - (xu)'\phi\}\phi^{m-1}u.
\end{aligned}$$

From lemma 1, we obtain

$$[\phi^m u, xu] = (m-1)\phi^{m-1}u + \phi^{m-1}u = m\phi^{m-1}u = mK_{m-1}.$$

Lemma 6.

$$[\phi^m u, \phi^n xu] = mK_{m+n-1} \qquad m = 1, 2, \ldots; n = 0, 1, 2, \ldots . \qquad (2.15)$$

Proof. (2.15) holds for $n = 0$, by virtue of lemma 5.

Set $[\phi^m u, \phi^{n-1}xu] = mK_{m+n-2}$. Then using the property of strong symmetry, we obtain

$$\begin{aligned}
[\phi^m u, \phi^n xu] &= [K_m \phi^n xu] = K'_m[\phi^n xu] - (\phi^n xu)'[K_m] \\
&= K'_m[\phi^n xu] - \phi'[K_m]\phi^{n-1}xu - \phi(\phi^{n-1}xu)'[K_m] \\
&= K'_m[\phi^n xu] - [K'_m, \phi]\phi^{n-1}xu - \phi(\phi^{n-1}xu)'[K_m] \\
&= K'_m[\phi^n xu] - K'_m[\phi^n xu] + \phi K'_m[\phi^{n-1}xu] - \phi(\phi^{n-1}xu)'[K_m] \\
&= \phi[K_m, \phi^{n-1}xu] = \phi m K_{m+n-2} = mK_{m+n-1}.
\end{aligned}$$

Lemma 7. Let $F_{n,m}=(\phi^{n-1}xu)'[\phi^m xu]$. Then

$$\phi'[\phi^{n-1}xu]\phi^{m-1}xu-F_{n,m}=-\phi F_{n,m-1}+\phi^{n+m-2}xu. \tag{2.16}$$

Proof. Using (2.5) over and over again we have

$$F_{n,m}=\sum_{j=2}^{n}\phi^{j-2}\phi'[\phi^m xu]\phi^{n-j}xu+\phi^{n-1}(xu)'[\phi^m xu]. \tag{2.17}$$

Using relations (2.8) and (2.9) we obtain

$$\begin{aligned}\phi'[\phi^{n-1}xu]\phi^{m-1}xu-F_{n,m}&=\phi'[\phi^{n-1}xu]\phi^{m-1}xu-\phi'[\phi^m xu]\phi^{n-2}xu\\ &\quad-\sum_{j=3}^{n}\phi^{j-2}\phi'[\phi^m xu]\phi^{n-j}xu-\phi^{n-1}(xu)'[\phi^m xu]\\ &=\phi\phi'[\phi^{n-2}xu]\phi^{m-1}xu-\phi\phi'[\phi^{m-1}xu]\phi^{n-2}xu-\phi\phi'[\phi^m xu]\phi^{n-3}xu\\ &\quad-\sum_{j=4}^{n}\phi^{j-2}\phi'[\phi^m xu]\phi^{n-j}xu-\phi^{n-1}(xu)'[\phi^m xu]\\ &=-\phi\phi'[\phi^{m-1}xu)\phi^{n-2}xu+\phi^2\phi'[\phi^{n-3}xu]\phi^{m-1}xu-\phi^2\phi'[\phi^{m-1}xm]\phi^{n-3}xu\\ &\quad-\sum_{j=4}^{n}\phi^{j-2}\phi'[\phi^m xu]\phi^{n-j}xu-\phi^{n-1}(xu)'[\phi^m xu]\\ &=\ldots=-\phi\sum_{j=2}^{n}\phi^{j-2}\phi[\phi^{m-1}xu]\phi^{n-j}xu+\phi^{n-1}\phi'[xu]\phi^{m-1}xu\\ &\quad-\phi^{n-1}(xu)'[\phi^m xu]\\ &=-\phi F_{n,m-1}+\phi^n(xu)'[\phi^{m-1}xu]+\phi^{n-1}\phi'[xu]\phi^{m-1}xu-\phi^{n-1}(xu)'[\phi^m xu]\\ &=-\phi F_{n,m-1}+\phi^{n-1}\{\phi(xu)'+\phi'[xu]-(xu)'\phi\}\phi^{m-1}xu\\ &=-\phi F_{n,m-1}+\phi^{n+m-2}xu.\end{aligned}$$

Therefore lemma 7 has been proved.

Lemma 8.

$$[\phi^m xu, xu]=m\phi^{m-1}(xu)\qquad m=1,2,\ldots. \tag{2.18}$$

Proof. When $m=1$, from lemma 1 we have

$$\begin{aligned}[\phi xu, xu]&=(\phi xu)'[xu]-(xu)'[\phi xu]\\ &=\phi'[xu]xu+\phi(xu)'[xu]-(xu)'[\phi xu]\\ &=\{\phi'[xu]+\phi(xu)'-(xu)'\phi\}xu=xu.\end{aligned}$$

Therefore (2.18) holds for $m=1$.

Assume

$$[\phi^{m-1}xu, xu]=(m-1)\phi^{m-2}(xu).$$

This yields

$$\begin{aligned}[\phi^m xu, xu]&=\phi'[xu]\phi^{m-1}xu+\phi(\phi^{m-1}xu)'[xu]-(xu)'[\phi^m xu]\\ &=\{\phi'[xu]+\phi(xu)'-(xu)'\phi\}\phi^{m-1}xu-\phi(xu)'\phi^{m-1}xu+\phi(\phi^{m-1}xu)'[xu]\\ &=\phi^{m-1}xu+\phi[\phi^{m-1}xu, xu]\\ &=\phi^{m-1}xu+\phi(m-1)\phi^{m-2}(xu)\\ &=m\phi^{m-1}xu.\end{aligned}$$

Lemma 9.

$$[\phi^m xu, \phi^n xu] = (m-n)\phi^{m+n-1}(xu) \qquad m=1,2,\ldots;\ n=0,1,2,\ldots. \tag{2.19}$$

Proof. When $m=0$, by virtue of lemma 8, (2.19) holds for any n. We assume

$$[\phi^{m-1}xu, \phi^n xu] = (m-n-1)\phi^{m+n-2}(xu).$$

Then, using (2.5), (2.9) and (2.16) we obtain

$$\begin{aligned}
[\phi^m xu, \phi^n xu] &= \phi'[\phi^n xu]\phi^{m-1}xu + \phi(\phi^{m-1}xu)'[\phi^n xu] - \phi'[\phi^m xu]\phi^{n-1}xu \\
&\quad - \phi(\phi^{n-1}xu)'[\phi^m xu] \\
&= \phi\{\phi'[\phi^{n-1}xu] - \phi'[\phi^{m-1}xu]\phi^{n-1}xu + (\phi^{m-1}xu)'[\phi^n xu] \\
&\quad - (\phi^{n-1}xu)'[\phi^m xu]\} \\
&= \phi\{\phi'[\phi^{n-1}xu] - F_{n,m}\} - \phi\{\phi'[\phi^{m-1}xu]\phi^{n-1}xu - (\phi^{m-1}xu)'[\phi^n xu]\} \\
&= \phi\{-\phi F_{n,m-1} + \phi^{n+m-2}xu\} - \phi\{\phi'[\phi^{m-1}xu]\phi^{n-1}xu - \phi'[\phi^n xu]\phi^{m-2}xu \\
&\quad - \phi(\phi^{m-2}xu)'[\phi^n xu]\} \\
&= \phi^{n+m-1}xu + \phi\{-\phi(\phi^{n-1}xu)'[\phi^{m-1}xu] - \phi'[\phi^{m-1}xu]\phi^{n-1}xu \\
&\quad + \phi'[\phi^n xu]\phi^{m-2}xu + \phi(\phi^{m-2}xu)'[\phi^n xu]\} \\
&= \phi^{n+m-1}xu + \phi[\phi^{m-1}xu, \phi^n xu] \\
&= \phi^{n+m-1}xu + (m-n-1)\phi^{m+n-1}xu = (m-n)\phi^{m+n-1}xu.
\end{aligned}$$

Lemma 10. Consider the equation

$$u_t = K(u). \tag{2.20}$$

We have the following results.

(i) If ϕ is a strong symmetry for (2.20) and ϕ is invertible, then ϕ^{-1} is also a strong symmetry for (2.20).

(ii) If ϕ is a hereditary symmetry and ϕ is invertible, then ϕ^{-1} is also a hereditary symmetry.

(iii) If ϕ is a strong symmetry for (2.20), then ϕ^2 is also a strong symmetry for (2.20).

(iv) If ϕ is a hereditary symmetry, the ϕ^2 is also a hereditary symmetry.

All these results can be verified by definition (see Zhu 1986).

3. Basic theorems

Let

$$\tau_0^m = mtK_{m-1} + xu \tag{3.1}$$

and

$$\tau_n^m = \phi^n \tau_0^m = mtK_{m+n-1} + \phi^n xu. \tag{3.2}$$

We now prove the following three theorems.

Theorem 1.

$$(\tau_n^m)_t = K'_m[\tau_n^m] \qquad m=1,2,\ldots;\ n=0,1,2,\ldots. \tag{3.3}$$

i.e. for any n, τ_n^m are symmetries for equation (2.1).

Proof. Since ϕ is a strong symmetry of the equation (2.1), ϕ maps the symmetry of equation (2.1) into the symmetry of (2.1), (see Fuchssteiner and Fokas 1981) and it is sufficient to prove the following identity:

$$(\tau_0^m)_t = K'_m[\tau_0^m]. \tag{3.4}$$

(3.4) can be proved as follows:

$$(\tau_0^m)_t = mK_{m-1} + mtK_{m-1,t} + (x\boldsymbol{u})_t.$$

By lemmas 4 and 5, we have

$$\begin{aligned} K'_m[\tau_0^m] &= mtK'_m[K_{m-1}] + K'_m[x\boldsymbol{u}] \\ &= mtK'_{m-1}[K_m] + K'_m[x\boldsymbol{u}] \\ &= mtK_{m-1,t} + [K_m, x\boldsymbol{u}] + (x\boldsymbol{u})'[K_m] \\ &= mtK_{m-1,t} + mK_{m-1} + (x\boldsymbol{u})_t. \end{aligned}$$

From the above two identities we know that (3.4) holds.

Theorem 2.

$$[K_m, \tau_n^l] = mK_{m+n-1} \qquad l=1,2,\ldots;\ m=1,2,\ldots;\ n=0,1,2,\ldots. \tag{3.5}$$

This theorem can be proved by lemmas 4 and 6.

Theorem 3.

$$[\tau_l^m, \tau_n^m] = (l-n)\tau_{l+n-1}^m \tag{3.6}$$

Proof. By definition of τ_n^m and by lemmas 6 and 9, we obtain

$$\begin{aligned} [\tau_l^m, \tau_n^m] &= mt[K_{m+l-1}, \phi^n x\boldsymbol{u}] + mt[\phi^l x\boldsymbol{u}, K_{m+n-1}] + [\phi^l x\boldsymbol{u}, \phi^n x\boldsymbol{u}] \\ &= mt(m+l-1)K_{m+n+l-2} - mt(m+n-1)K_{m+n+l-2} + (l-n)\phi^{l+n-1}x\boldsymbol{u} \\ &= mt(l-n)K_{m+n+l-2} + (l-n)\phi^{l+n-1}x\boldsymbol{u} \\ &= (l-n)\tau_{l+n-1}^m. \end{aligned}$$

We now consider the following non-linear equation which corresponds to the new symmetries τ_n^m:

$$u_t = \tau_n^m. \tag{3.7}$$

In Li (1982), we considered the eigenvalue problem

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}-\mathrm{i}\xi & q\\ r & \mathrm{i}\xi\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \tag{3.8}$$

with

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_t = \begin{pmatrix}A & B\\ C & -A\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}. \tag{3.9}$$

In the non-isospectral case, i.e.

$$\xi_t = \sum_{j=0}^{N} k_j(t)\xi^{N-j} \tag{3.10}$$

the compatibility condition

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_{xt} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}_{tx}$$

yields the non-linear evolution equation

$$\begin{pmatrix}q_t\\ \gamma_t\end{pmatrix} = 2\sum_{j=0}^{N} k_{N-j}(t)(\tfrac{1}{2})^j\phi^j x\boldsymbol{u} + 2\sum_{j=0}^{N}\alpha_{N-j}(t)(\tfrac{1}{2})^j\phi^j\boldsymbol{u}. \tag{3.11}$$

Thus we have the following theorem.

Theorem 4. Equation (2.1),

$$u_t = \phi^m\boldsymbol{u}$$

and equation (3.7),

$$u_t = \tau_n^m = mt\phi^{m-n-1}\boldsymbol{u} + \phi^n x\boldsymbol{u}$$

are special cases of equation (3.11).

In fact, when $N = m$, $k_i = 0$, $i = 1, 2, \ldots, m$ (i.e. $\xi_t = 0$) and $\alpha_0 = 2^{m-1}$, $\alpha_i = 0$, $i \neq 0$, (3.11) is reduced to (2.1). When $N = n + m - 1$, $\alpha_0 = mt2^{m+n-2}$, $\alpha_i = 0$, $i \neq 0$, $k_{m-1} = 2^{n-1}$, $k_{j=0}$, $j \neq m-1$ and (3.11) is reduced to (3.7).

4. The reduction of equation (2.1)

Equation (2.1), which we considered above, is a two-dimensional vector equation. By a reduction technique, it can be reduced to a scalar equation.

4.1. NLS hierarchy

Taking $r = \varepsilon q^*$, $\varepsilon = \pm 1$, we see that two equations of (2.1) are compatible. Take one of them and let

$$\phi = \frac{1}{\mathrm{i}}[-D + 2\varepsilon qD^{-1}q^* + 2\varepsilon qD^{-1}q(\ )^*]. \tag{4.1a}$$

Equation (2.1) becomes

$$q_t = K_m = \phi^m(-\mathrm{i}q). \tag{4.2a}$$

The first three K_i are as follows:

$$K_0 = -\mathrm{i}q \qquad K_1 = q_x \qquad K_2 = \mathrm{i}[q_{xx} - 2\varepsilon|q|^2q]. \tag{4.3a}$$

Define

$$\tau_n^m = mtK_{m+n-1} + \phi^n(-\mathrm{i}xq) \qquad m = 1, 2, \ldots;\ n = 0, 1, 2 \ldots. \tag{4.4a}$$

Then we have the following proposition.

Proposition 1.

(i) τ_n^m are symmetries for equation (4.2*a*) (4.5*a*)

(ii) $[K_m, K_n]=0$ (4.6*a*)

(iii) $[K_m, \tau_n^l]=mK_{m+n-1}$ (4.7*a*)

(iv) $[\tau_l^m, \tau_n^m]=(l-n)\tau_{l+n-1}^m$. (4.8*a*)

4.2. MKdV hierarchy

$r=\varepsilon q$, $\varepsilon=\pm 1$, q is a real function of x, t. Let

$$\phi_1=-D^2+4\varepsilon q_x D^{-1}q+4\varepsilon q^2. \tag{4.1b}$$

Consider equations

$$u_t=\phi^{2m+1}\boldsymbol{u}. \tag{4.2b}$$

Since

$$\phi\boldsymbol{u}=\begin{pmatrix} q_x \\ \varepsilon q_x \end{pmatrix} \qquad \phi^2\begin{pmatrix} \nu \\ \varepsilon\nu \end{pmatrix}=\begin{pmatrix} \phi_1\nu \\ \varepsilon\phi_1\nu \end{pmatrix}$$

equation (4.2*b*) can be read as

$$\begin{pmatrix} q \\ \varepsilon q \end{pmatrix}_t=\phi^{2m}\phi\boldsymbol{u}=\phi^{2m}\begin{pmatrix} q_x \\ \varepsilon q_x \end{pmatrix}=\begin{pmatrix} \phi_1^m q_x \\ \varepsilon\phi_1^m q_x \end{pmatrix}$$

which can be reduced to scalar equations:

$$q_t=\phi_1^m q_x=\tilde{K}_m \tag{4.2c}$$

where

$$\tilde{K}_0=q_x \qquad \tilde{K}_1=-q_{xxx}+6\varepsilon q^2 q_x. \tag{4.3b}$$

Since ϕ^{2m} are hereditary symmetries and are strong symmetries for the equation $\begin{pmatrix} q \\ \varepsilon r \end{pmatrix}_t=\phi\boldsymbol{u}$, from lemma 10, we conclude that ϕ_1^m are hereditary symmetries and are strong symmetries for the equation $q_t=q_x$.

Define

$$\tilde{\tau}_1^m=(2m+1)t\tilde{K}_m+(xq)_x \quad \tilde{\tau}_n^m=\phi_1^{n-1}\tilde{\tau}_1^m=(2m+1)t\tilde{K}_{m+n-1}+\phi_1^{n-1}(xq)_x. \tag{4.4b}$$

we obtain (note that $\tilde{K}_m$ and $\tilde{\tau}_n^m$ defined here are K_{2m+1} and τ_{2n-1}^{2m+1} in theorem 2, respectively) the following proposition.

Proposition 2

(i) $\tilde{\tau}_n^m$ are symmetries of equation (4.2*c*) (4.5*b*)

(ii) $[\tilde{K}_m, \tilde{K}_n]=0$ (4.6*b*)

(iii) $[\tilde{K}_m, \tilde{\tau}_n^l]=(2m+1)\tilde{K}_{m+n-1}$ (4.7*b*)

(iv) $[\tilde{\tau}_l^m, \tilde{\tau}_n^m]=2(l-n)\tilde{\tau}_{l+n-1}^m$. (4.8*b*)

4.3. Sine-Gordon and sinh-Gordon hierarchies

We note that the operator ϕ_1 given by (4.1b) is invertible, i.e. the operator ϕ^2 is invertible. We are going to give an explicit formula for ϕ_1^{-1}.

Let $\phi_1 g = h$, where g and h are arbitrary functions in U. Then when $\varepsilon = -1$ we have (see the appendix)

$$g = C \sin 2D^{-1}q - \tfrac{1}{2}\int_0^v \sin 2(v-u)\left[\frac{D^{-1}h(u)}{q}\right]_u \mathrm{d}u = \phi_i^{-1}h \qquad (4.9a)$$

where the constant C and function v are given by

$$C = -\tfrac{1}{2}\lim_{x\to-\infty} D^{-1}h/q \qquad (4.10)$$

and

$$v = D^{-1}q. \qquad (4.11)$$

When $\varepsilon = 1$, we obtain (see the appendix)

$$g = C \sinh 2D^{-1}q - \tfrac{1}{2}\int_0^V \sinh 2(v-u)\left[\frac{Dh(u)}{q}\right]_u \mathrm{d}u = \phi_1^{-1}h. \qquad (4.9b)$$

We point out that when $m = -1$, equation (4.2c) reduces to the scalar equation $q_t = \phi_1^{-1}q_x$. When $\varepsilon = -1$, we take $g = q_t$ in (4.9a) and $h = q_x$ in (4.10) and the equation $q_t = \phi_1^{-1}q_x$ is reduced to the sine-Gordon equation

$$q_t = -\tfrac{1}{2}\sin 2D^{-1}q. \qquad (4.12)$$

When $\varepsilon = 1$, we take $g = q_t$ in (4.9b) and $h = q_x$ in (4.10) and the equation $q_t = \phi_{1q_x}^{-1}$ is reduced to the sinh-Gordon equation

$$q_t = -\tfrac{1}{2}\sinh 2D^{-1}q. \qquad (4.13)$$

Define $\tilde{\tau}_1^m$ and $\tilde{\tau}_n^m$ as (4.4b), since ϕ_1 is invertible and m and n can be taken as positive or negative. From the results (i) and (ii) in lemma 10, we conclude that ϕ_1 and ϕ_1^{-1} are hereditary symmetries and are strong symmetries of of $q_t = \phi_1^m q_x$ ($m = 0, \pm1, \pm2 \ldots$) and especially they are strong symmetries of the MKdV equation $q_t = \phi_1 q_x$ and the sine-Gordon (or sinh-Gordon) equation $q_t = \phi_1^{-1}q_x$.

In this case (4.5b)-(4.8b) in proposition 2 also hold, but l, m and n can be negative. By the way, we note that in proposition 2, when $m = 1$ and n is taken to be positive, the $\tilde{\tau}_n'$ defined here are the so-called τ symmetries in Chen *et al* (1982b), but the symmetries to be obtained here are more than in Chen *et al* (1982a, b, 1983) because n can be taken as negative here.

Since ϕ_1 is invertible, we conclude that the MKdV and sine-Gordon hierarchies (or the sinh-Gordon hierarchy) have the same symmetries.

Acknowledgments

For several reasons the authors are greatly indebted to the referees. First of all, because their constructive comments improved this paper considerably. Secondly because they pointed out the important reference to Strampp and Oevel (1985) in which the DNLS equation's new symmetries and Lie algebra were found.

One of the authors (LYS) would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

This work is partially supported by the Science Fund of the Chinese Academy of Sciences.

Appendix

We prove that (4.9a) is an explicit formula of the inverse operator ϕ_1^{-1}, where the operator ϕ_1 is given by (4.1b).

Rewrite the operator ϕ_1 as follows:

$$\phi_1 = D(-D + 4\varepsilon q D^{-1} q). \tag{A1}$$

Let

$$\phi_1 g = h$$

which yields

$$-g_x + 4\varepsilon q D^{-1} q g = D^{-1} h. \tag{A2}$$

Put

$$v = D^{-1} q = \int_{-\infty}^{x} q(x)\,\mathrm{d}x. \tag{A3}$$

Then

$$q = v_x.$$

Considering g as a function of v, $g(q(x)) = g(v(x))$, we have

$$g_x = g_v v_x = g_v q. \tag{A4}$$

Substituting (A4) into (A3) we obtain

$$-g_v + 4\varepsilon D^{-1} v_x g = D^{-1} h / q$$

or

$$-g_v + 4\varepsilon \int_0^v g\,\mathrm{d}v = D^{-1} h / q \tag{A5}$$

which yields

$$-g_{vv} + 4\varepsilon g = H(v) \tag{A6}$$

where

$$H(v) = \left[\frac{D^{-1} h(v)}{q(v)}\right]_v. \tag{A7}$$

The solution of equation (A6) can be obtained as follows.

(i) $\varepsilon = -1$, the general solution of equation (A6) is

$$g = C_1 \sin 2v + C_2 \cos 2v - \tfrac{1}{2} \int_0^v \sin 2(v - u) H(u)\, du \tag{A8}$$

where C_1, C_2 are arbitrary constants.

It is naturally required that $g \to 0$ as $x \to -\infty$ so putting $x \to -\infty$ in (A3) we can get $C_2 = 0$.

C_1 is determined by (A5). Putting $x \to -\infty$ in (A5) we have

$$-g_v(0) = \lim_{v \to 0} D^{-1}h/q = \lim_{x \to -\infty} D^{-1}h/q.$$

On the other hand, differentiating (A8) with respect to v and putting $x \to -\infty$ yields

$$g_v(0) = 2C_1.$$

Thus

$$C_1 = -\tfrac{1}{2} \lim_{x \to -\infty} D^{-1}h/q. \tag{A9}$$

Substituting (A9) into (A8), we obtain (4.9*a*).

(ii) $\varepsilon = +1$, equation (A6) becomes

$$-g_{vv} + 4g = H(v)$$

the general solution of which is

$$g = C_1 \sinh 2v + C_2 \sinh 2v - \tfrac{1}{2} \int_0^V \sinh 2(v-u) H(u) \,\mathrm{d}u. \tag{A10}$$

Then in the same way we determine that $C_2 = 0$ and C_1 is given by (A9).

References

Chen H H, Lee Y C and Lin J E 1982a *Advances in Nonlinear Waves* vol II, ed L Debnath (Boston, MA: Pitman Advanced Publishing Program) p 233

—— 1982b *Phys. Lett.* **91A** 381

—— 1983 *Physica* D **9** 439

Fokas A S and Fuchssteiner B 1981 *Phys. Lett.* **86A** 341

Fuchssteiner B and Fokas A S 1981 *Physics* **4D** 47

Li Y-S 1982 *Sci. Sin.* A **5** 911

Li Y-S and Zhu G-C 1985 *Scientia Sinica* to be submitted

Oevel W and Fuchssteiner B 1982 *Phys. Lett.* **88A** 323

Olver P J 1980 *Math. Proc. Camb. Phil. Soc.* **88** 71

Strampp W and Oevel W 1985 *Prog. Theor. Phys.* **74** 922L

Zhu G-C 1986 *Kexue Tongbao* **31** 92